## 6.1 Psychology of Learning Mathematics

Learning is often defined as a relatively permanent change in behavior as a result of experience. One of the greatest contributions psychology has made to humanity is its explanation of how people learn—how we come to know what we know, and how we change our knowledge and behavior in response to experience. Decades of psychological research have established what good teachers already know about the most effective ways to reach students and help them master new material.

The psychology of learning mathematics is concerned with the various types of learning and the reasons and causes that allow learning to occur. There are different theories deriving from the psychology of learning and they are

**a) Kinesthetic Learning**

Kinesthetic learning incorporates both physiological and cognitive processes facilitating internalizing new knowledge for later recall (learning).

**b) Constructivist Learning**

Assimilating, restructuring and incorporating new information into prior knowledge involves a constructivist learning process.

**c) Memory Consolidation**

Memory consolidation is a process when knowledge is internalized, stabilized and isolated for later recall.

**d) Collaborative Learning**

Working to cultivate skills as a team, group or social setting; collectively participating toward a common theme is collaborative learning

The purpose of learning of mathematics is to understand and remember things that are learned. A single unified theory of learning is almost impossible because learning is such a complex and conjectural process. Experiences and researches have given mathematics educators some ideas of the learning behaviors and thinking patterns of different learners.

The essence of understanding a mathematical concept is to have a mental representation or a schema that can reflect the structure of that concept. Construction of mental representations can be facilitated by the external representations such as words, symbols, static diagrams, dynamic animations, sounds and smell, which will be converted to learner's internal representations and finally reach the stage of memory retention.

A learner needs to solve mathematical problems independently once he or she understands certain concepts, theorems and masters some specific rules. Problem solving skills are needed for performing different tasks whether the problems look familiar to the learner or not. The question here is: Does the learner really know how to apply the mathematical knowledge to simplify a polynomial, to evaluate a trigonometric expression, to use factor theorem in solving a polynomial equation, to prove points A, B, C, D are concyclic or to obtain a set of linear inequalities from a word problem of optimization?

Although it is impossible to videotape the thinking processes inside the learner's head, indirect methods do exist to examine the degree of conceptual or procedural understandings. By sharing solutions to problems and ideas, students' thinking can be empowered in learning mathematics

Abstract mathematical concepts like functions and probability theories are always hard to teach and learn unless they can be represented in some understandable or perhaps interesting ways. Therefore imagery thinking is perhaps the most effective way of understanding and constructing the conceptual hierarchies with learner's own words and images as well as self-questioning.

Knowing how to solve a mathematics problem is very different to understanding how to solve it. The former is mechanical and the latter shows some kinds of mathematical thinking. A learner must be highly engaged in the process of mathematics learning in order to achieve a high level of mastery. Behavioral engagement (observable behavior together with readiness for particular type of behavior), cognitive engagement (knowledge of an object, conception, world view) and affective engagement (emotions, beliefs, values, attitudes, ethics, morals) have positive effects on learner's ability in solving mathematics problems and eventually the mathematics achievement.

Learning mathematics can be thought as a social interaction process rather than an individual one as pointed out by Vygotsky who says communication is a cultural tool. By explaining and justifying their thinking, learners create their own knowledge and develop mathematical meanings using their language of the culture which will be transformed into mathematical concepts later. When students are actively involved both physically and mentally, they make connections from their ordinary language to mathematical language. By carefully designing appropriate activities for students, mathematics teachers can create the zone of proximal development (ZPD) and consequently every student can develop understanding of the concepts established culturally.

## 6.2 Gagne's Types of Learning and their Appropriateness for Learning Mathematics

Gagne has enunciated the following eight types of learning which have great relevance for the learning of mathematics

1. Signal learning
2. Stimulus-response learning
3. Chaining
4. Verbal Association
5. Discrimination learning
6. Concept learning
7. Rule learning and
8. Problem Solving

### 6.2.1 Signal Learning

Signal Learning refers to learning to respond to the signal. In this type of learning the individual learns to make a general diffuse response to a signal. This is the classical conditioned response of Pavlov. This type is here called "Signal learning" to emphasize that the learner is associating an already available response with a new stimulus or "Signal".

There is a wide acquaintance with signal learning whether in common domestic animals or other human beings. Most people have observed one or more instances of signal learning in household pets; for e.g. the cat or dog may run to the kitchen when he hears his food disk placed on the floor.

#### The Phenomenon of Signal Learning

The learning of Pavlov's dog to salivate in response to a signaling buzzer is an example of signal learning. However the "eye blink" is an example of a signal learning situation that has been extensively studied in human beings.

When a small puff of air is delivered to the cornea of a person's eye, the eye blinks. This is the connection Pavlov called as unconditioned reflex, meaning that the action is there to begin with and is not conditioned on any previous learning. Now if a click sounded about half second before the puff of air reaches the cornea we have one of the important conditions for the establishment of a learned connection.

The click (or other neutral stimulus) is called the conditioned stimulus (the "Signal"). When this sequence of events click puff of air is repeated a few times, it is usually possible to demonstrate the existence of a newly learned connection namely

$$\underset{\text{Click}}{S} \longrightarrow \underset{\text{blink}}{R}$$

This is done by presenting the click by itself, without the puff of air and noting that the blink response occurs.

Experimental studies have shown that the conditioned blink is not the same as the unconditioned blink (Kimble-1961). The unconditioned blink is a more rapid response (it occurs 0.05 to 0.1 second) whereas the signaled blink takes 0.25 to 0.5 second. Thus it appears that what is learned may be called an anticipatory blink to a signal. Such a blink does not avoid the puff of air. But the learned blink anticipates the puff of air; it signals "puff of air to come". The pairing of signal and unconditioned stimulus have to be repeated a number of times in order to establish a stable response. Signal-response connections have been shown to occur in a very few trials when the signal accompanies a stimulus arousing a strong emotion.

**Conditions for Signal Learning:**

*i)* There must be a natural reflex-typically of a reflexive conditioned response (fear, anger, pleasure) on the part of the learner. (It must be clear that an unconditioned stimulus can evoke such an unconditioned response. Individual differences determine the rapidity with which people acquire signal-response connections.)

*ii)* *Contiguity:* The signaling stimulus and the unconditioned stimulus must be presented in close proximity to each other.

*iii)* *Repetition:* Repetition of the paired stimuli is also necessary. The connection appears to increase in strength as the repetition of paired stimuli increase in number.

### 6.2.2 Stimulus-Response Learning

Thorndike called it as trial and error learning. It is also called operant learning (Skinner).Many writers call it instrumental learning and this name has two advantages:

- It emphasizes the precise skilled nature of the response involved as in "using instruments".
- It implies that the learned connection is instrumental in satisfying some motive.

The phrase "Stimulus-Response learning" is chosen to emphasize two other characteristics:

i) Such learning concerns a single connection between a stimulus and a response, not multiple or chained associations.

ii) The stimulus and the response appear to become integrally bound together in such learning, in a way that does not happen with signal learning. Example: Infants' learned response of holding the nursing bottle in the proper position for feeding. By gradually "helping", the child holds the bottle properly and gradually removing the help so that the child's own responses are appropriate, the stimulus response connection becomes established. Eventually the child, once given the bottle can reinstate the necessary response.

This learning event appears to be not so much a trial and error procedure as a successive approximation procedure. The learning that occurs here as Skinner says, is a matter of 'Shaping'.

**Conditions for Stimulus-Response learning**

i) There must be a stimulus and a terminating act which provides reinforcement. It is necessary that the infant sucks on the nipple and there by fills the mouth with food. This produces what Thorndike called "a satisfying state of affairs". In other words taking of food by sucking is the reinforcer that is made contingent upon the performance of holding the bottle. For learning to occur, one needs only to ensure that there is such a response, terminating in reinforcement, which can be performed by the learner.

ii) *The empirical law of effect:* The learning of any new S-R connection is dependent upon directly following consequences. This condition is called empirical law of effect.

iii) *Contiguity:* Contiguity has an important part to play in the establishment of S-R connection. The shorter the time elapsing between the occurrence of the learned response and the occurrence of reinforcement, the more rapidly learning will take place.

iv) *Repetition:* Repetition of the stimulus situation is also a necessary condition for the occurrence of stimulus-response learning.

### 6.2.3 Chaining

Chaining is the connection of a set of non verbal individual stimulus-response connections in sequence. Some sequences such as tuning on a television set, or a washing machine are made up of motor responses. Other responses are entirely verbal. For the acquisition of sequences that are non-verbal, the term "chaining" will be used. The sub variety involving verbal behavior is called verbal association or verbal sequence learning. A child learning to open the door with a key is a classical example of chaining.

**Conditions for Chain Learning**

i) Each individual stimulus-response connection should have been previously learned. One cannot expect a chain such as opening a door with a key to be learned in an optimal way unless the learner is already able to carry out the S-R's that constitute the chain. The learner must be able to

   a) identify the key's upright position

   b) insert it into the lock fully

   c) turn it clockwise fully

   d) push the door and open.

   If a child complains that he is not able to open the door with key after being shown how, one immediately suspects that he has not fully mastered one of the above mentioned links. Failure to learn on a simple occasion usually indicates that one or more links have not been previously learnt.

ii) Assuring that the links are known, the first condition for the establishment of a chain is setting the learner to reinstate them one after the other in the proper order.

   A **Prompting** method may be used in which additional external cues are employed to ensure the reinstatement of the link in the proper order.

iii) *Contiguity:* The individual links in the chain must be executed in close succession, if the chain is to be established.

iv) A next condition relates to repetition. Practice is almost always desirable.

Condition of the reinforcement is present in the learning of chaining. The terminal links must provide reinforcement.

In mathematics, teaching of geometrical constructions is a typical example of chaining (non-verbal) as it follows a series of logical sequences or steps ((S-R) links).

### 6.2.4 Verbal Association

Verbal association is the connection of a set of verbal S-R connection in sequence.

Probably the simplest verbal chains are illustrated in the activity of naming. A youngster is told while being shown a three dimensional object "This shape is called a tetrahedron". If conditions are otherwise right, the next time he sees this particular object, he will be able to say that it is a 'tetrahedron'. But these other conditions are important. It is fairly easy to identify some conditions which would ensure that the learner would not be able to say "tetrahedron". To mention two of the most obvious

- He may not have discriminated the object as a stimulus
- He may not have learnt to say its name.

The act of naming a specific object like a tetrahedron seems likely to be a chain of at least two links. The first of these is an observing response, a S-R in which the learner identifies the triangular character of its sides, and distinguishes it from other three dimensional objects of roughly the same size and color. The second link in the S-R connection that enables the individual to stimulate himself to say "tetrahedron" is an integral representation of the object. As a diagram a simple set of naming is represented by

| S ⟶ R | | s ⟶ R | |
|---|---|---|---|
| Object | observation | internal representation | tetrahedron |

The small 's' in the diagram stands for the internal representation of the object resulting from observation.

**Condition for learning Verbal Sequences**

i) Each link of the chain to be acquired must have been previously learned as an $S \rightarrow R$.

ii) Mediating connections between each verbal unit and the next must have been previously learned. The greater the learner's available supply of these "coding" connections, the more rapidly will learning take place.

iii) The verbal units must be presented in proper sequence.

iv) The learner must actively make the responses required to learn the chain.

v) Length of the chain to be limited to 7±2

vi) Confirmation of correct responses must be provided from the learning situation.

## 6.2.5 Discrimination Learning

Discrimination learning may occur when an individual is confronted with a set of stimulus objects. In some circumstances of discrimination learning, the learner may have to acquire a response which differentiates by name or otherwise the stimulus features of a single member of a set from those of other members or she may learn to distinguish several different members of a set, making a different response to each.

Acquiring discrimination is obviously an undertaking of great importance in everyday life and in school learning. As for learning in the school, the student is confronted with the necessity of learning discrimination among printed colors, shapes, letters, numerals, etc. Discrimination learning leads to perceptual differentiation within five media namely objects, space, events, representation and symbols. The child's earliest learned perceptions pertains to objects and space. Discrimination of events comes a bit later or as the child becomes able to manipulate objects and to move about. Still later and much emphasized in early school grades, is the learning of discriminations of representation (pictures) and symbols.

Discrimination learning is often concerned with the distinctive features of stimulus objects. Thus the child learns to respond differently to the characteristics that serve to distinguish objects from one another-shape, sizes, and colors and so on. This may be the simplest form of S-R form.

Consider the example-the discrimination of a circle and an ellipse. Initially two figures may be presented on a set of cards, some containing two circles, other two ellipses and still others a circle and an ellipse. As a response is made to each card the children are told whether it is correct or incorrect. They have acquired the discrimination, when it becomes evident that they 'see' the difference in figures without appreciable errors.

**Conditions for Learning Discrimination**

i) The learner should have the ability to recall and reinstate the different response chains necessary to exhibit the discrimination.

ii) The learner must be provided with selective reinforcement of correct versus incorrect responses to the stimulus.

*Example:* Children learning to distinguish closed and open plane figures.

a) Identifying 'closed' figures when shown pairs (closed & open) in which they point the correct figure.

b) A large set of figures, closed and open may be shown in which the children must point to either the closed or open ones.

c) Closed and open figures may be shown one by one and the children may be able to say "closed" or "open". Differential reinforcement will be employed. Some indication for correct and incorrect responses should be provided.

When multiple discriminations are learned the learning task becomes more difficult because of confusions among the stimuli to be discriminated.

## 6.2.6 Concept Learning

In this type the learner is able to respond in a single way to a collection of objects as a class. This class extends beyond the members that were originally present. This kind of learning is called concept learning. Having previously learned to make distinction the learner may be required to acquire the capability of responding to the set of stimuli as a class and distinguishing members of the class form non-members; this is what is meant by learning a concept. For example when a learner is presented with a class of stimulus objects {triangle, circle, rectangle, pentagon} and if he is able to respond to the entire class with a single response as 'plane figures', the learner has learnt the concept of 'plane figures'. The concept learning is complete only when the learner is able to give new instances of the concept and distinguishes examples from non-examples.

The following are the different stages by which a learner develops a concept.

- **Discrimination stage**

Discriminations are prerequisite to the learning of concepts. Assume that a learner has to acquire the concept of a 'straight line', first he has to distinguish (discriminate) a straight line from the lines which are not straight.

(curved). Thus the learner first masters the discrimination so that he responds positively to the straight line (discrimination) and negatively (in some different fashion) to the curved line when presented as a pair

*Example:* 

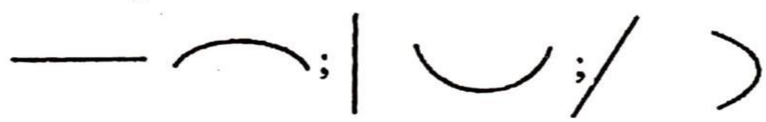
## Generalization stage

In this stage the same straight line is paired with the varied dimensions of curved lines. None of these are straight, but they differ in their irrelevant dimensions such as amount of curvature, orientation etc. Contrast practice is continued until the learner convincingly discriminates straight lines from "non-straight" line. This results in generalization of the original discrimination learning.

*Example:* 

## Abstraction stage

In this stage there are variations in the irrelevant dimensions of the non-straight lines and also in the irrelevant dimension of the straight lines. Once the learning of this stage has been mastered, there is convincing proof that the learner has "abstracted" the relevant object quality. He has acquired the concept of line's 'straightness'.

*Example:* 

To make a final check on the learning of the concept, it is customary to choose a new example which has not been used during learning new straight line. We say that the concept has been acquired when the learner can identify such a novel (situation) or instance. In other words he is able to classify examples from non-examples.

## Learning concrete concepts

The term concept has many meanings. We should consider the most fundamental meaning of the term concept which is exhibited in individual behavior by responding to objects, qualities such as those implied by names, 'red', double 'circular' and 'smooth' or learning common objects such as 'chair' , 'tree' and 'house'. It is customary to denote these as concrete concepts, since they can be denoted by being pointed out. In other words they are concepts by observation. Hence learning of these concepts depends upon direct observation. Here the people respond to the objects in terms of some common observable property, in a real sense people classify them.

## Conditions for Learning Concrete Concepts

i) *Discrimination:* This must be recalled by the learner at the time of concept learning. When verbal instructions are employed, the verbal lables must have previously been learnt as associates to the stimulus feature that represent the concept.

ii) The specific stimulus object, or stimulus dimension, is presented as an instance along with a non-instance (different object in dimension). The learner identifies the dimension by name.

iii) In the generalization stage, instruction introducces the instance along with various non-instances.

iv) The stimulus feature which is to become the concept is itself varied and presented along with various non-instances.

v) Conditions of reinforcement is presented in the concept learning situation (confirmation of the correct response)

vi) Contiguity is also another factor. As for repetition it does not appear to be necessary when other conditions are optimal.

## Learning defined concepts

Many concepts cannot be learnt as concrete concepts. Instead they must be learned by definition and accordingly may be called defined concepts. These are abstract in the sense that they involve relations (concepts of square root, prime numbers etc.). Sometimes they are called as abstract concepts to distinguish from concrete concepts. For example the concept of a diagonal is a defined concept and not a concrete concept. The statement "a diagonal is a line connecting opposite corners of a quadrilateral" represents a relation (connected) between the two concepts" ("opposite corners of a quadrilateral").

The defined concept is in a formal sense one type of a rule, the rule that classifies the events or objects. The set of prime numbers is classified by defining prime numbers as "any number divisible only by one and by itself".

It may be noted that some of the concepts like circles, ellipse may have been learned previously as concrete concepts. This is not an unusual occurrence. However the concrete form of a concept such as circle has some inadequacies when compared with its defined form. For instance, the child who has attained the concrete concept of circle might identify plane figure which are not quite circlular as circle as the points on the curve might not be equidistant from a fixed point. Of course, the concrete concept of circle may be perfectly useful in many situations.

Some concepts can be learned only by their definitions. (example: probablility, limits). They have no concrete counter parts which can be identified by their appearance. This abstraction must be understood and communicated by means of the statement of a classifying rule which is also called a definition.

Definitions are statements that express rules for classifying. An individual who has acquired defined concept has learned this kind of a rule and is able to apply it to any instance of the class. It is not essential that the individual learns the rule statement itself in order to state that he has learned the concept. The definition mainly represents the classifying rule he has learned.

Defined things whether they are things or relation must apparently be represented by definitions composed of other concepts. These components are already familiar to the learner which have been previously learned. This means some concepts must originally be learned without definitions(they are concrete concepts).If the concept of pentagon must be learned from a definition containing the concepts "straight and side" the latter are likely to have been learned as concrete concepts at some earlier time.

**Condition for Learning Defined Concepts**

i) Defined concepts are actually learned by means of their definitions. In contrast to concrete concepts defined concepts cannot be readily acquired simply by the presentation of a variety of instances and non-instances, whose characteristics can be perceived by the learner.

ii) The learner should know the definition, meaning of terms, adjectives, adverbs etc.

iii) The definition should be presented in an oral or printed form.

## 6.2.7 Rule Learning

A defined concept is a particular kind of rule, a rule that classifies. Thus a rule must be an internal state which governs one's behavior and enables one to demonstrate a relationship. A rule must be more than the verbal statement by which it is represented. For instance, the prepositions, "things equal to something are equal to each other". It must be something that accounts for regularity of behavior. A rule is an inferred capability that enables an individual to respond to a class of stimulus situation with a class of performance, such performances being predictably related to the stimuli by a specific class of relations. An individual respond, to a class of stimulus situation (2+3, 3+4, 3+5) with a class of performance (3+2, 4+3, 5+3) that are predictably related to the stimuli by a relation that may be expressed as independence of order. The rule that governs this behavior may be represented by the statement "Adding two numbers 'a' and 'b' is independent of the order in which 'a' & 'b' are combined".

Typically a rule is composed of several concepts. An individual who possesses the rule, as a capability can be observed to identify these component concepts and also to demonstrate that they relate to one another in the particular manner of the rule. Individual must have learned these component concepts as prerequisites to learning the rule. Assuming that these concepts have been acquired, learning a rule becomes a matter of learning their correct sequence.

Rules are obviously of many types in so far as their content is concerned. They may be defined concepts, serving the purpose of distinguishing among different ideas, and they maybe capability which enable the individual to respond to specific situation by applying classes of relations. Evidently rules might vary in such properties as abstractness and complexity.

**Condition for Rule Learning**

i) The learner should have learnt the component concepts of the rule

ii) The learner should have verbal ability and language skill

iii) The rule should be presented in an oral or printed form.

iv) An activity requiring the learner to demonstrate the rule should be presented.

## 6.2.8 Problem Solving

Problem Solving is an extention of rule learning .Problem solving requires an individual to discover a combination of previously learned rules to apply to solve a novel problem. Problem solving combines two or more rules to produce a new capability, resulting in the formation of a higher order rule. Higher order rules are learning strategies which enable individuals to solve other problems of a similar type and such higher-order rules often result from the learner's thinking in a problem-solving situation.

Problem solving becomes associated with both intellectual skills and cognitive strategies.

## Condition for Problem Solving

i) The rules must be previously learned by the learner

ii) The learner should have verbal ability and language skill to read and understand the problem.

iii) The learner should be able to recall and apply the appropriate rules

iv) The learner must use cognitive strategies to solve the problem.

Among all the types of learning enunciated by Gagne, the last five types have greater relevance for the learning of mathematics as the capabilities expected of a student of mathematics are to

- name objects such as triangles, rectangles, squares etc., (verbal association)
- discriminate the objects by means of specific attributes (discrimination learning)
- define the concept and classify the instances and non instances (concept learning)
- relate the concepts in a particular sequence to form rules (rule learning) and
- select and apply the relevant rules for solving mathematical problems (problem solving)

This could be well illustrated by the following example,

In $\Delta$ ABC, <B = 90°, AC = 5 cms. AB = 12 cms., find AB.

To solve the above problem the following types of learning are involved.

1. $\Delta$ ABC is a right angled triangle – Verbal association
2. Distinguishing between the three sides as hypotenuse, and the sides containing the right angle. – Discrimination learning
3. Recalling the concepts of hypotenuse, right angled triangle etc. – Concept learning
4. Relating the three sides by Pythagoras theorem – Rule learning
5. Applying Pythagoras theorem to find AB – Problem solving.

It is therefore possible to draw a linear hierarchy of types of learning which can be applied to learning of mathematics.

## 6.3 The Ideas of Piaget and their Appropriateness for Learning Mathematics

Jean Piaget's work on children's cognitive development, specifically with quantitative concepts, has acquired much attention within the field of mathematics education. Piaget explored children's cognitive development to study his primary interest in genetic epistemology. One contribution of Piagetian theory concerns the developmental stages of children's cognition. His work on children's quantitative development has provided mathematics educators with insights into how children learn mathematical concepts and ideas.

Piaget believed that the development of a child occurs through a continuous transformation of thought processes. A developmental stage consists of a period of months or years when certain development takes place. Although students are usually grouped by chronological age, their development levels may differ significantly as well as the rate at which individual children pass through each stage. This difference may depend on maturity, experience, culture, and the ability of the child. Piaget believed that children develop steadily and gradually throughout the varying stages and that the experiences in one stage form the foundations for movement to the next.

### 6.3.1 Stages of Cognitive Development

Piaget has identified four primary stages of development: sensorimotor, preoperational, concrete operational and formal operational.

#### 6.3.1.1 Sensorimotor Stage

This stage is characterized by the progressive acquisition of object permanence in which the child is able to find objects after they have been displaced, even if the objects have been taken out of his field of vision.

An additional characteristic of children at this stage is their ability to link numbers to objects (e.g., one dog, two cats, three toys). To develop the mathematical capability of a child in this stage, the child's ability might be enhanced if he is allowed to act on the environment in unrestricted (but safe) ways in order to start building concepts. Children at the sensorimotor stage have some understanding of the concepts of numbers and counting. There should be a solid mathematical foundation by providing activities that incorporate counting and thus enhance children's conceptual development of number.

### 6.3.1.2 Preoperational Stage

The characteristics of this stage include an increase in language ability, symbolic thought, egocentric perspective, and limited logic. In this second stage, children should engage with problem-solving tasks that incorporate available materials such as blocks, sand, and water. While the child is working with a problem, the teacher should elicit conversation from the child. The verbalization of the child, as well as his actions on the materials, gives a basis that permits the teacher to infer the mechanisms of the child's thought processes.

There is lack of logic associated with this stage of development; rational thought makes little appearance. The child links together unrelated events, sees objects as possessing life, does not understand point-of-view, and cannot reverse operations. For example, a child at this stage who understands that adding two to five yields seven cannot yet perform the reverse operation of taking two from seven.

Children's perceptions in this stage of development are generally restricted to one aspect or dimension of an object at the expense of the other aspects. For example, Piaget tested the concept of conservation by pouring the same amount of liquid into two similar containers. When the liquid from one container is poured into a third, wider container, the level is lower and the child thinks there is less liquid in the third container. Thus the child is using one dimension, height, as the basis for his judgment of another dimension, volume.

While teaching students in this stage of development teachers should employ effective questioning about characterizing objects. For example, when students investigate geometric shapes, a teacher could ask students to group the shapes according to similar characteristics.

### 6.3.1.3 Concrete Operational Stage

The third stage is characterized by remarkable cognitive growth, when children's development of language and acquisition of basic skills accelerate dramatically. Children at this stage utilize their senses in order to know; they can now consider two or three dimensions simultaneously. For example, in the liquids experiment, if the child notices the lowered level of the liquid, he also notices the dish is wider, seeing both dimensions at the same time. Additionally, seriation and classification are the two logical operations that develop during this stage and both are essential for understanding number concepts. Seriation is the ability to order objects according to increasing or decreasing length, weight, or volume. On the other hand, classification involves grouping objects on the basis of a common characteristic.

According to Burns & Silbey (2000), "hands-on experiences and multiple ways of representing a mathematical solution can be ways of fostering the development of this cognitive stage." The importance of hands-on activities cannot be over emphasized at this stage. These activities provide students an avenue to make abstract ideas concrete, allowing them to get their hands on mathematical ideas and concepts as useful tools for solving problems.

Because concrete experiences are needed, teachers might use manipulatives with their students to explore concepts such as place value and arithmetical operations. Existing manipulative materials include: pattern blocks, Cuisenaire rods, algebra tiles, algebra cubes, geoboards, tangrams, counters, dice, and spinners. However, teachers are not limited to commercial materials, they can also use convenient materials in activities such as paper folding and cutting. As students use the materials, they acquire experiences that help lay the foundation for more advanced mathematical thinking. Furthermore, students' use of materials helps to build their mathematical confidence by giving them a way to test and confirm their reasoning.

One of the important challenges in mathematics teaching is to help students make connections between the mathematics concepts and the activity. Children may not automatically make connections between the work they do with manipulative materials and the corresponding abstract mathematics: children tend to think that the manipulations they do with models are one method for finding a solution and pencil-and-paper math is entirely separate. For example, it may be difficult for children to conceptualize how a four by six inch rectangle built with wooden tiles relates to four multiplied by six, or four groups of six. Teachers could help students make connections by showing how the rectangles can be separated into four rows of six tiles each and by demonstrating how the rectangle is another representation of four groups of six.

Providing various mathematical representations acknowledges the uniqueness of students and provides multiple paths for making ideas meaningful. Creating opportunities for students to present mathematical solutions in multiple ways (e.g., symbols, graphs, and tables) is one tool for cognitive development in this stage.

### 6.3.1.4 Formal Operational Stage

The child at this stage is capable of forming hypotheses and deducing possible consequences, allowing the child to construct his own mathematics.

Furthermore, the child typically begins to develop abstract thought patterns where reasoning is executed using pure symbols without the necessity of perceptive data. For example, the formal operational learner can solve $x + 2x = 9$ without having to refer to a concrete situation presented by the teacher, such as, "Ram ate a certain number of chocolates. His sister ate twice as many. Together they ate nine. How many did Ram eat?"

Reasoning skills in this stage include clarification, inference, evaluation, and application.

- *Clarification:* Clarification requires students to identify and analyze elements of a problem, allowing them to interpret the information needed in solving a problem. By encouraging students to take out relevant information from a problem statement, teachers can help students enhance their mathematical understanding.
- *Inference:* Students at this stage are developmentally ready to make inductive and deductive inferences in mathematics.
- *Evaluation:* Evaluation involves using criteria to judge the adequacy of a problem solution.
- *Application:* Application involves students connecting mathematical concepts to real-life situations.

Piaget believed that the amount of time each child spends in each stage varies by environment. All students in a class are not necessarily operating at the same level. Teachers could benefit from understanding the levels at which their students are functioning and should try to ascertain their students' cognitive levels to adjust their teaching accordingly. By emphasizing methods of reasoning,the teacher provides critical direction so that the child can discover concepts through investigation. The child should be encouraged to self-check, approximate, reflect and reason while the teacher studies the child's work to better understand his thinking.

Piaget outlined several principles for building cognitive structures. During all developmental stages, the child experiences his/her environment using whatever mental maps he/she has constructed so far. If the experience is a repeat one, it fits easily into the child's cognitive structure (that is it is ***assimilated*** into the existing cognitive structure) so that the child maintains mental ***equilibrium***. If the experience is different or new, the child loses equilibrium (hence ***disequilibrium***), and alters his/her cognitive structure to ***accommodate*** the new conditions. In this way, the child builds more and more adequate cognitive structures.

As children develop, they progress through stages characterized by unique ways of understanding the world. During the sensorimotor stage, young children develop eye-hand coordination schemes and object permanence. The preoperational stage includes growth of symbolic thought, as evidenced by the increased use of language. During the concrete operational stage, children can perform basic operations such as classification and serial ordering of concrete objects. In the final stage, formal operations, students develop the ability to think abstractly and meta cognitively, as well as reason hypothetically. In general, the knowledge of Piaget's stages helps the teacher understand the cognitive development of the child as the teacher plans stage-appropriate activities to keep students active.

## Brunner's Discovery Learning

Discovery learning, a concept **advocated by Jerome Bruner**, is at the essence of how students learn concepts and ideas. Bruner talked about the "act of discovery" as if it were a performance on the part of the student.

Discovery Learning is a learning method that encourages students to ask questions and formulate their own tentative answers, and to deduce general principles from practical examples or experiences.

Discovery Learning is a learning situation in which the principal content of what is to be learned is not given but must be independently discovered by the student.

Discovery learning can be defined simply as a learning situation in which the principal content of what is to be learned is not given, but must be independently discovered by the learner, making the student an active participant in his learning.

Jerome Bruner lays out **two targets** for discovery learning theory:

1. Discovery Learning Theory should act as a refined extension of the broad based theory constructivism by focusing on the individual.
2. Discovery Learning Theory should serve as a way of defining and providing structure to the way in which individuals learn thus acting as a guide for educational research.

There are **four components** to the Discovery Learning Theory:

1. **Curiosity and uncertainty**
   Bruner felt that experiences should be designed that will help the student be willing and able to learn. He called this the predisposition toward learning. Bruner believed that the desire to learn and to undertake problem solving could be activated by devising problem activities in which students would explore alternative solutions. The major condition for the exploration of alternatives was "the presence of some optimal level of uncertainty."This related directly to the student's curiosity to resolve uncertainty and ambiguity. According to this idea, the teacher would design discrepant event activities that would pique the students' curiosity.
2. **Structure of knowledge**
   Bruner expressed it by saying that the curriculum specialist and teacher "must specify the ways in which a body of knowledge should be structured so that it can be most readily grasped by the learner." This idea became one of the important notions credited to Bruner. He explained it this way: "Any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form."
3. **Sequencing**
   Instruction should lead the learner through the content in order to increase the student's ability to "grasp, transform and transfer" what is learned. In general sequencing should move from enactive (hands-on, concrete), to iconic (visual), to symbolic (descriptions in words or mathematical symbols). However, this sequence will be dependent on the student's symbolic system and learning style. As we will see later, this principle of sequencing is common to theories developed by Piaget, as well as other cognitive psychologists
4. **Motivation**
   Bruner suggests that movement from extrinsic rewards, such as teacher's praise, toward intrinsic rewards inherent in solving problems or understanding the concepts is desirable and also learning depends upon knowledge of results when it can be used for correction. Feedback to the learner is critical to the development of knowledge. The teacher can provide a vital link to the learner in providing feedback at first, as well helping the learner develop techniques for obtaining feedback on his or her own.

There are **three principles** associated with Discovery Learning Theory:

1. Instruction must be concerned with the experiences and contexts that make the student willing and able to learn (readiness).
2. Instruction must be structured so that it can be easily grasped by the student (spiral organization).
3. Instruction should be designed to facilitate extrapolation and or fill in the gaps (going beyond the information given).

Bruner identified **six indicators** or benchmarks that revealed cognitive growth or development:

1. Responding to situations in varied ways, rather than always in the same way.
2. Internalizing events into a 'storage system' that corresponds to the environment.
3. Increased capacity for language.
4. Systematic interaction with a tutor (parent, teacher, or other role model).
5. Language as an instrument for ordering the environment.
6. Increasing capacity to deal with multiple demands.

**Advantages and Disadvantages of Discover Learning**

**Advantages**

- Supports active engagement of the learner in the learning process
- Fosters curiosity
- Enables the development of lifelong learning skills
- Personalizes the learning experience
- Highly motivating as it allows individuals the opportunity to experiment and discover something for themselves
- Builds on learner's prior knowledge and understanding
- Develops a sense of independence and autonomy
- Make them responsible for their own mistakes and results
- Learning as most adults learn on the job and in real life situations
- A reason to record their procedure and discoveries - such as not repeating mistakes, a way to analyze what happened, and a way to record a victorious discovery
- Develops problem solving and creative skills
- Finds new and interesting avenues of information and learning - such as gravy made with too much cornstarch can become a molding medium

These sorts of arguments can be regrouped in two broad categories

- Development of Meta cognitive skills (including some higher level cognitive strategies) useful in lifelong learning.
- Motivation

**Disadvantages**

- (Sometimes huge) cognitive overload, potential to confuse the learner if no initial framework is available, etc.
- Measurable performance (compared to hard-core instructional designs) is worse for most learning situations.
- Creations of misconceptions ("knowing less after instruction")
- Weak students have a tendency to "fly under the radar" (Aleven et al. 2003) and teacher's fail to detect situations needing strong remediation or scaffolding.
- Some studies admit that strong students can benefit from weak treatments and others conclude that there is no difference, but more importantly they also conclude that weak students benefit strongly from strong treatments.

## Critical Analysis of Mathematics Curriculum at the secondary level (state Board) based on principles and organization of Mathematics curriculum and NCF 2005

Mathematics is one of the oldest fields of knowledge and study and has long been considered one of the central components of human thought. Some call it a science, others an art and some have even likened it to a language. It appears to have pieces of all three and yet is a category by itself.

According to the National Curriculum Framework (NCF) 2005, the main goal of Mathematics education in schools is the 'mathematisation' of a child's thinking. Clarity of thought and pursuing assumptions to logical conclusions is central to the mathematical enterprise. While there are many ways of thinking, the kind of thinking one learns in Mathematics is an ability to handle abstractions and an approach to problem solving.

The NCF envisions school Mathematics as taking place in a situation where:

1. Children learn to enjoy Mathematics rather than fear it
2. Children learn “important” Mathematics which is more than formulas and mechanical procedures
3. Children see Mathematics as something to talk about, to communicate through, to discuss among themselves, to work together on

4. Children pose and solve meaningful problems
5. Children use abstractions to perceive relationships, to see structures, to reason out things, to argue the truth or falsity of statements
6. Children understand the basic structure of Mathematics: arithmetic, algebra, geometry and trigonometry, the basic content areas of school Mathematics, all of which offer a methodology for abstraction, structuration and generalisation
7. Teachers are expected to engage every child in class with the conviction that everyone can learn Mathematics.

On the other hand, the NCF also lists the **challenges facing** Mathematics education in our schools as:

1. A sense of fear and failure regarding Mathematics among a majority of children
2. A curriculum that disappoints both a talented minority as well as the non-participating majority at the same time.
3. Crude methods of assessment that encourage the perception of Mathematics as mechanical computation - problems, exercises, methods of evaluation are mechanical and repetitive with too much emphasis on computation
4. Lack of teacher preparation and support in the teaching of Mathematics
5. Structures of social discrimination that get reflected in Mathematics education often leading to stereotypes like 'boys are better at Mathematics than girls. However the difficulty is that computations become significantly harder, and it becomes that much more difficult to progress in arithmetic.

The NCF, therefore, recommends:

1. Shifting the focus of Mathematics education from achieving 'narrow' goals of mathematical content to 'higher' goals of creating mathematical learning environments, where processes like formal problem solving, use of heuristics, estimation and approximation, optimisation, use of patterns, visualisation, representation, reasoning and proof, making connections and mathematical communication take precedence
2. Engaging every student with a sense of success, while at the same time offering conceptual challenges to the emerging Mathematician
3. Changing modes of assessment to examine students' mathematisation abilities rather than procedural knowledge
4. Enriching teachers with a variety of mathematical resources. A major focus of the NCF is on removing fear of Mathematics from children's minds. It speaks of liberating school Mathematics from the tyranny of the one right answer found by

applying the one algorithm taught. The emphasis is on learning environments that invite participation, engage children, and offer a sense of success.

**Methods of Learning**

The NCF says that many general tactics of problem solving can be taught progressively during the different stages of school: abstraction, quantification, analogy, case analysis, reduction to simpler situations, even guess-and-verify exercises, is useful in many problem-solving contexts.

Moreover, when children learn a variety of approaches (over time), their toolkit becomes richer, and they also learn which approach is the best? Children also need exposure to the use of heuristics, or rules of thumb, rather than only believing that Mathematics is an 'exact science'. The estimation of quantities and approximating solutions is also an essential skill.

Visualization and representation are skills that Mathematics can help to develop. Modelling situations using quantities, shapes and forms are the best use of Mathematics. Mathematical concepts can be represented in multiple ways, and these representations can serve a variety of purposes in different contexts.

For example, a function may be represented in algebraic form or in the form of a graph. The representation 'p/q' can be used to denote a fraction as a part of the whole, but can also denote the quotient of two numbers, 'p' and 'q.' Learning this about fractions is as important, if not more, than learning the arithmetic of fractions. There is also a need to make connections between Mathematics and other subjects of study. When children learn to draw graphs, they should also be encouraged to think of functional relationships in the sciences, including geology. Children need to appreciate the fact that Mathematics is an effective instrument in the study of science.

The importance of systematic reasoning in Mathematics cannot be over-emphasised, and is intimately tied to notions of aesthetics and elegance so dear to Mathematicians. Proof is important, but in addition to deductive proof, children should also learn when pictures and constructions provide proof. Proof is a process that convinces a skeptical adversary; school Mathematics should encourage proof as a systematic way of argumentation. The aim should be to develop arguments, evaluate arguments, make and investigate conjectures, and understand that there are various methods of reasoning.

The NCF also speaks of mathematical communication – that it is precise and employs unambiguous use of language and rigour in formulation, which are important characteristics of mathematical treatment. The use of jargon in Mathematics is deliberate, conscious and stylised. Mathematicians discuss what appropriate notation is since good notation is held in high esteem and believed to aid thought. As children grow older, they should be taught to appreciate the significance of such conventions and their use. This would mean, for instance, that setting up of equations should get as much coverage as solving them.

**Organization of the Curriculum,** The NCF recommends the following for secondary stage of schooling:

Students now begin to perceive the structure of Mathematics as a discipline. They become familiar with the characteristics of mathematical communication: carefully defined terms and concepts, the use of symbols to represent them, precisely stated propositions, and proofs justifying propositions. These aspects are developed particularly in the area of geometry.

Students develop their facility with algebra, which is important not only in the application of Mathematics, but also within Mathematics in providing justifications and proofs. At this stage, students integrate the many concepts and skills that they have learnt into a problem-solving ability. Mathematical modelling, data analysis and interpretation taught at this stage can consolidate a high level of mathematical literacy. Individual and group exploration of connections and patterns, visualisation and generalisation, making and proving conjectures are important at this stage, can be encouraged through the use of appropriate tools that include concrete models as in Mathematics laboratories and computers.

**On Assessment**, the NCF recommends that Board examinations be restructured, so that the minimum eligibility for a State certificate is numeracy, reducing the instance of failure in Mathematics. At the higher end, it is recommended that examinations be more challenging, evaluating conceptual understanding and competence. The NCF's vision of excellent mathematical education is based on the twin premise that all students can learn Mathematics and that all students need to learn Mathematics. It is, therefore, imperative that Mathematics education of the very highest quality is offered to all children.

**For Example: Criticism to the New Mathematics Curriculum of State of Kerala**

With the introduction of New curriculum a lot of criticism has come from various corners especially through media and it has become a centre of discussion .Some of the criticisms felt by the investigator through document analysis of NCF 2005, KCF 2007 and the Mathematics textbooks from standard VIII to X are following below.

1. There is an urgent need for continuous programme of monitoring and evaluation of the curriculum.
2. The curriculum is disappointing not only to the nonparticipating majority, but also to the talented minority by not offering them challenges.
3. The curriculum is overloaded and high emphasis is given on knowledge aspects.
4. The curriculum and textbooks don't reflect the needs and aspirations of the learner.
5. The ongoing curriculum doesn't help to achieve various categories of objectives in a fair manner.
6. It doesn't give due importance to differentially abled children.
7. Teachers don't get chances to participate in regular evaluation of curriculum.
8. Curriculum is delinked with daily life skills.
9. Teachers and teacher educators felt lack of involvement in the process of curriculum construction.
10. The major defect of the school curriculum is lack of practical knowledge, emphasis on information rather than understanding, and it embodies a heavy load of subject matter.
11. The preparation of secondary school curriculum is highly centralised at the regional government respectively to limited experts.
12. The syllabus is very vast and is expected to be covered at the end of each academic year which forces the teacher to proceed whether the students understand or not.

**Suggestions to minimize the limitations in the curriculum**

- Changes in curriculum should not be something synonymous with a change of government.
- The regional education bureau should design a mechanism whereby teachers opinions are included in the preparation of the syllabus and text books. Through this involvement, problems related to syllabus such as its vastness, relevance to daily life of the children, redundancy etc can be solved.
- In order to realise educational objectives, the curriculum should be conceptualised as a structure that articulates required experiences.
- The curriculum should be prepared with the participation of teachers, students, professionals and concerned bodies.

- Curriculum must be flexible enough for the teacher to deal with the capacity of each individual child.
- Effort should be made to bring attitudinal changes among students, teachers and parents towards learning of Mathematics.
- As teachers are those who are doing the actual work of transacting the syllabus, they should be consulted for their opinions on what the syllabus should focus and how the text books should be prepared.
- Curriculum should be designed in such a way that the students and society can use it in their daily life and students should be aware of its practical application.

**Conclusion**

It is clear from the present study that, the existing condition of mathematics learning owes much to the deformities in the present curriculum. The text book preparation should be done strictly based on the guidelines of NCF 2005 and the curriculum committee should take necessary steps to reform the mathematics curriculum by incorporating suggestions from various stakeholders, so that great change can take place in future in the field of mathematics education and miserable condition of students' hatredness towards mathematics can be banished a lot.