1) Meaning – Problems, Problem Solving and Problem Posing:

- **Problem** is a topic, event or activity, which no memorized or specified rules are known regarding its **solution**.
- **Problem posing** is a crucial component for mathematics discipline and
- **Problem solving** is the situation of elimination of confusion in human mind.

2) Characteristics of a Good Problem

The following are the characteristics of a good problem in mathematics.

- The problem should be real rather than an artificial one.
- It should facilitate the integration of old and new processes.
- The solution of the problem should result in learning new higher order rules.
- The solution of the problem should help in transfer of knowledge.
- The problem should be educationally significant, productive of important and worthwhile learning.
- It should be possible of a solution. The students should be equipped with background information and skills which are prerequisite for solving the given problem.
- It should be related to the sub-unit, the unit and the course.
- It should form the basis for further learning.
- It should be clear and free from ambiguities.
- It should be interesting and challenging.
- It should arouse the curiosity of the students.
- It should occur frequently in life situations.
- It should provide best mental discipline to the students.
- It should have both practical and social values.
- It should be neither too difficult nor too easy for the students.
- It should facilitate realisation of the objectives of teaching mathematics.

8.1.7.5 Reasons for Difficulties in Solving Problems

- Lack of interest and motivation.
- Lack of language clarity in understanding the problem.
- Inability to analyse the problem thoroughly.
- Lack of focus on the key relationships.
- Inability to identify the interrelationship among the given data.
- Lack of fluency in the mental visualisation or diagrammatic representation of the problem.
- Inability to recall and apply appropriate rules and formulae.
- Lack of systematic and orderly written work.
- Lack of skill and practice in solving problems.
- Lack of proficiency in the fundamental arithmetic operations.
- Inadequate knowledge of fundamental mathematical concepts, rules and formulae.
- Difficulty in reading, identifying and using mathematical symbols.

8.1.7.6 Types of Mathematical Problems

Mathematical problems are of *four* types.

- *Puzzle Problem:* These are problems designed for the exercise of ingenuity and patience, as these problems create some bewilderment or perplexity in the individual who faces it. Sometimes people solve them as a leisure time activity merely for the sake of joy and pleasure that they derive. However, such problems preserve the curiosity of the student and he feels joy in solving them.
- *Catch Problem:* These problems display a jugglery of words. Such problems check the mental alertness of the students, but have little bearing on training mental faculties.
- *Real Problems:* These problems are directly related to the real life experiences of the students. They emerge from the real life situations. The solution of such problems help the students in facing future life problems with ease and confidence. The solution of such problems stimulates the curiosity and help in training the mental faculties of the children.
- *Unreal Problems:* Problems which are beyond the purview of real life situations are called unreal problems. Such problems give false information to the students.

3) Problem-solving Strategies and steps in Problem Solving - Gagne's views on Problem Solving

8.1.7 Problem-Solving Method

The problem-solving method is one which involves the use of the process of problem-solving or reflective thinking or reasoning. Problem-solving method, as the name indicates, begins with the statement of a problem that challenges the students to find a solution. The problem centres around the subject matter under study and requires the use of information and skills available to the students. In the process of solving the problem the students may be required to gather data, analyse and interpret the information, to arrive at a solution to the problems.

Definitions of Problem-Solving

Problem solving presupposes the existence of a problem in the teaching-learning situation. A problem is an obstruction of some sort to the attainment of an objective, a sort of difficulty which does not enable the individual to reach a goal easily.

"A problem occurs in a situation in which a felt difficulty to act is realised. It is a difficulty that is clearly present and recognised by the thinker. It may be a purely mental difficulty or it may be physical and involve the manipulation of data. The distinguishing thing about a problem, however, is that it impresses the individual who meets it as needing a solution. He recognises it as a challenge". – ***Yokam and Simpson***

"Problem solving is a planned attack upon a difficulty or perplexity for the purpose of finding a satisfactory solution". – ***Risk, T.M***

"Problem solving is an educational device whereby the teacher and the pupils attempt in a conscious, planned, purposeful manner to arrive at an explanation or solution to some educationally significant difficulty".

– ***James Ross***

From the above definitions, problem solving involves the following.

- a goal to be reached
- a felt difficulty to reach the goal
- challenging the felt difficulty through conscious, planned and purposeful attack
- reaching the goal or arriving at a satisfactory solution to the problem at hand

Therefore, as used in teaching-learning situation, problem-solving is a method in which the felt difficulty to act in an educational situation is realised and then an attempt is made in a conscious and purposeful way to find its solution.

Main Objectives of Problem-Solving Method

The main objective of problem-solving method is to stimulate the reflective and creative thinking of the students. It involves the thought process that results from a doubt, a perplexity or a problem. The approach leads to the formulation of generalisations that are useful in future situations involving the solution of similar problems. The solution of a problem, whatever be its nature, practical or informational involves the process of reflective thinking.

8.1.7.1 Steps in Problem-Solving

Problem-solving follows definite and specific steps.

Identifying and defining the problem

The problem arises out of a felt need and out of existing student activities and environment activities. The students should be able to identify and clearly define the problem. The problem that has been identified, should be interesting, challenging and motivating for the students to participate in exploring.

Analysing the problem

The problem should be carefully analysed as to what is given and what is to be found out. Given facts must be identified and expressed, if necessary in symbolic form. The relationships are to be clearly stated. Relations that are not explicitly stated may be supplied by the students.

Formulating tentative hypothesis

The focus at this stage is on hypothesising – searching for a tentative solution to the problem. Analysis of the given data, and analysis of interrelationships among the given facts help the students in formulating hypothesis or educated guesses as the solution to the problem at hand.

Testing the hypothesis

Appropriate methods should be selected to test the validity of the tentative hypothesis as a solution to the problem. If it is not proved to be the solution, the students are asked to formulate alternate hypothesis and proceed.

Checking the result or Verification of the result

At this step the students are asked to determine their results and substantiate the expected solution. The students should be able to make generalisations and apply it to their daily life.

Gagne's views on Problem Solving

Problem Solving is an extention of rule learning .Problem solving requires an individual to discover a combination of previously learned rules to apply to solve a novel problem. Problem solving combines two or more rules to produce a new capability, resulting in the formation of a higher order rule. Higher order rules are learning strategies which enable individuals to solve other problems of a similar type and such higher-order rules often result from the learner's thinking in a problem-solving situation.

Problem solving becomes associated with both intellectual skills and cognitive strategies.

Condition for Problem Solving

i) The rules must be previously learned by the learner

ii) The learner should have verbal ability and language skill to read and understand the problem.

iii) The learner should be able to recall and apply the appropriate rules

iv) The learner must use cognitive strategies to solve the problem.

8.1.7.2 Approaches and Techniques to Problem-Solving

Problem-solving advocates the following approaches

- *Analytic and synthetic methods.*
- *Inductive and deductive methods.*

(For details refer to earlier parts of this chapter)

- *Method of analogies*

  In analogy, problems are solved by comparing them with similar problems that have been solved before. Thus the method of solution becomes explicit and clear.

- *Restatement Method*

  Problem solving becomes easier if the student is able to redefine the given problem using his own language and symbols. This approach is known as restatement method.

- *Method of Dependencies*

  In this method, the problem is solved by focussing on mutually dependent components in the problem. The analysis of the problem into its constituent elements throws light on the mutually dependent elements in the problem. The interrelationships among the elements can be made use of for reaching the correct solution of the problem.

- *Graphic Method*

  In this method, the problem is represented using diagrams and figures. The graphic representation aids the students in determining fundamental relationships that exist among the given data and to look for further details and relationships necessary for solving the given problem. This method is very helpful in proving theorems, solving riders, problems relating to mensuration. Pythagoras theorem, set theory, functions and relations etc.

8.1.7.3 Teacher's Role in Problem-Solving Method

The teacher plays a significant role in problem solving method. The teacher's role is to:

- ensure an atmosphere of freedom in the class.
- create the problem situation.
- assist the students in accepting, defining and stating the problem.
- help the students in analysing the problem and in breaking up the problem into simple units.
- help the students keep their attention focussed on the main problem all the time.
- guide the students in locating relevant source materials.
- encourage the students in seeking important relationships in the data.
- help the students develop an attitude of open mindedness and critical enquiry
- exhibit spirit of enquiry and discovery

8.1.7.7 Merits of Problems-Solving Method

- Problem-solving provides a real life like experience to the children.
- It develops in pupils good habits of planning, thinking, reasoning and independent work.
- It develops initiative and self-responsibility among the students.
- It takes into account individual differences.
- It helps the students to develop reflective thinking.
- It helps the students to approach future problems with confidence.
- It builds a mental attitude for effective learning based on critical thinking
- It helps the children develop mental traits of open-mindedness and tolerance as the children see many sides to a problem and listen to many points of view.

8.1.7.8 Demerits of Problem-Solving Method

- Not all students are problem solvers.
- The problem solving method becomes monotonous if used too frequently.
- It is time consuming and consequently it is not possible to cover the syllabus on time.
- The success of this method depends upon mathematics teachers who are well versed in critical thinking and reflective thinking. Not all mathematics teachers are well versed in these types of thinking.
- Reference and resource materials may be difficult to come by.
- Only a skilled and resourceful teacher will be able to make an effective use of this method.
- All topics in mathematics cannot be taught through this method.
- Textbooks are not available according to this method.
- Lack of interest and motivation on the part of the students can spoil the effectiveness of this method.

Conclusions

Problem-solving method can be an effective method for teaching mathematics in the hands of an able and resourceful teacher of mathematics.

4) Strategies of Mathematics Problem posing

Problem posing is closely associated with the problem solving method. Problem posing involves generating new problems and questions to explore about a given situation, as well as reformulating a problem during the course of solving the problem related to it. Teachers can help to developing this habit by understanding the children's thinking processes and developing these processes using generative questions. The problem posing method involves developing problem posing as an instructional intervention to improve problem solving skills and to improve disposition towards solving. Problem posing is an indicator of learning that takes place.

When we encourage children to be problem posers, we are inviting them to do what mathematicians do — that is, to look closely, seek patterns, offer conjectures, and set out on paths that are not clearly marked. In the process of their investigations, mathematicians also develop attitudes about learning, such as perseverance, willingness to revise their thinking, and appreciation for the value of risk taking.

Let us, as an example, consider the statement *4X 5 = 20.*

The first step of problem posing is to look closely or observe the statement critically. In the above statement following are some of the observations we can make.

There are two multipliers.

The two multipliers are two consecutive natural numbers.

One of the multipliers is even and other is odd.

One is a multiple of 2 and other is multiple of 5.

The product is 4 more than a square number (16) and 5 less than another square number (25).

The multipliers are consecutive counting numbers.

The difference between the multipliers is 1.

After making the observation on the statement *4X 5 = 20,* what possible problems can be posed? Some problems posed here as exemplars:

1. Do we always get an even product when multiplying an odd number by an even number?

2. What do we find if we multiply an odd number by an odd number? An even number by an even number? What if we multiplied three odd numbers or three even numbers?

3. What if we continued to multiply by multiples of 2 and multiples of 5? What patterns might we see?

4. What if we tried using multipliers that are the same to make 20? Is this result possible? What products are possible using multipliers that are the same?

5. What if we tried adding two numbers to equal 20? How many ways could we do so? What do we notice about odd and even numbers when adding to make 20?

6. Why, when we add an odd and an even number, do we get an odd number, but when we multiply an odd number by an even number we get an even number?

What are the benefits of problem posing for learning?

It develops the spirit of inquiry. The more we observe, the more we want to find out.

It leads the learner into unknown territory.

It requires and promotes reflective thinking especially during posing the problems.

It supports learners in asking the perennial question that mathematicians pose: Is this always true? That is, did this relationship occur fortuitously, or does a pattern lurk behind these numbers?

The other benefit is that problem posing involves searching for patterns.

Uncovering patterns is certainly joyful, but even more rewarding is discovering why those patterns are occurring. Let's see what we notice about our original problem: $4\times 5 = 20$.We see that 20 is 4 away from the nearest smaller square, 16, and is 5 away from the nearest larger square, 25. Why? We notice that 4×5 is $4 \times (4+1)$ or $(5-1) \times 5$

MATHEMATICAL PROBLEM POSING STRATEGIES

Mathematics teachers might use one or more strategies to formulate new problems or encourage their students in mathematics classes to be good problem posers as well as a good problem solvers. Strategies could be used depending on the most suitable conditions (mathematics content, students' levels, learning outcomes and mathematical thinking types). Problem posing situations are classified as free, semi-structured or structured situations.

Free Problem Posing Situations

Situations from daily life (in or outside school) can help a student to generate some questions leading him/her to construct a problem. Students are asked to pose a problem to encourage them to "make up a simple or difficult problem" or "construct a problem suitable for a mathematics competition

(or a test)" or "make up a problem you like." It is more useful if the teacher tries to relate the real life situations to the mathematics content being taught and to ask students to pose new problems. This will be more effective in developing students' mathematical thinking. Problem posing situations might take these types: every day life situation, free problem posing, problems they like, problems for a mathematics competition, problems written for a friend and problems generated for fun.

Semi-Structured Problem Posing Situations

Students are given an open-ended situation and are invited to explore it using knowledge, skills, concepts and relationships from their previous mathematical experiences and this can take the following forms:

Open-ended problems (i.e. mathematical investigation).

Problems similar to given problems.

Problems with similar situations.

Problems related to specific theorems.

Problems derived from given pictures.

Word problems.

This strategy was developed with student teachers as the following (Abu-Elwan, 1999):

1) A semi-structured situation from a student's daily life was presented to all students.
2) Students were asked to complete the situations using their perspective to be able to pose problems from that formed situation.

Students can generate problems by omitting the questions from given situations.

Structured Problem Posing Situation

Any mathematical problem consists of known (given) and unknown (required) data. The teacher can simply change the known and pose a new problem, or keep the data and change the required. Brown and Walter (1990, 1993) designed an instructional problem formulating approach based on the posing of new problems from already solved problems, but they have also recommended varying the conditions or goals of given problems.

This reformulation approach appears to be the most effective method for introducing structured problem posing activities in mathematics classrooms.

In order to create teaching/learning situations that provide a good problem posing situations, Lowrie (1999) recommended the mathematics teacher to:

1) encourage students to pose problems for friends who are at or near their own level until they become more competent in generating problems;
2) ensure that students work cooperatively in solving the problems so that the problem generator gains feedback on the appropriateness of the problems they have designed;
3) ask individuals to indicate the type of understanding and strategies the problem solver will need to use in order to solve the problem successfully before a friend generates a solution;
4) encourage problem solving teams to discuss, with one another, the extent to which they found problems to be difficult, confusing, motivating or challenging;
5) pProvide opportunities for less able students to work cooperatively with a peer who challenged the individual to engage in mathematics at a higher level than they were accustomed;
6) challenge students to move beyond traditional word problems by designing problems that are open ended and associated with real life experiences; and
7) encourage students to use technology (calculators, CDs, computers) in developing their mathematical thinking skills, so they can use this technology to generate new mathematical situations.

5) Divergent Thinking and Creativity in Mathematics

6.7 Divergent Thinking in Mathematics

Creative production is often characterized by the divergent nature of human thought and action. Divergence is usually indicated by the ability to generate many, or more complex or complicated, ideas from one idea or from simple ideas or triggers. Divergent Thinking is thinking outwards instead of inward. It is the ability to develop original and unique ideas and then come up with a problem solution or achieve an objective.

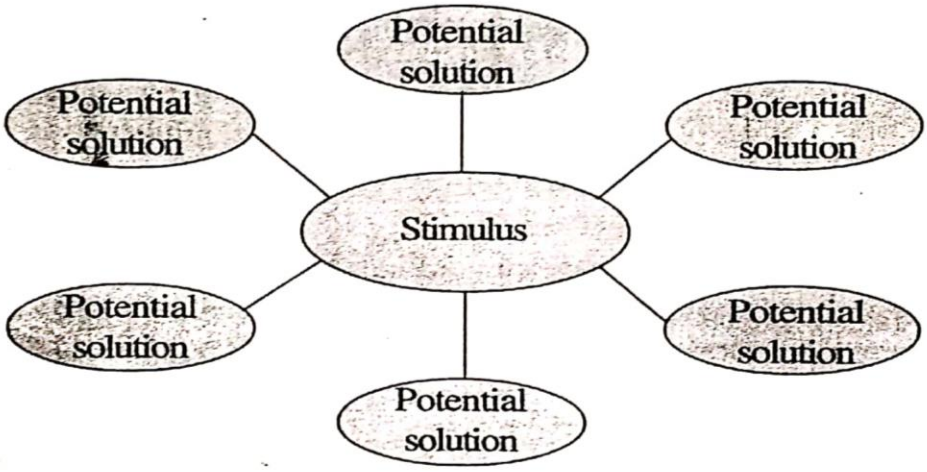

Fig.6.1: Divergent Thinking

Traditionally the eight elements below are ones commonly thought of as inherent elements of divergent thinking.

- **Fluency** - The ability to generate a *number of ideas* so that there is an increase of possible solutions or related products.
- **Flexibility** - The ability to produce different *categories or perceptions* whereby there are a *variety* of different ideas about the same problem or thing.
- **Elaboration** - The ability to *add to, embellish, or build off of* an idea or product.
- **Originality** - The ability to *create fresh, unique, unusual, totally new, or extremely different* ideas or products
- **Complexity** - The ability to conceptualize *difficult, intricate, many layered or multifaceted* ideas or products.

- **Risk-taking** - The willingness to be *courageous, adventuresome, daring* – trying new things or taking risks in order to stand apart.
- **Imagination** - The ability to *dream up, invent,* or to see, to think, to conceptualize new ideas or products to be ingenious.
- **Curiosity** - The trait of exhibiting *probing behaviors,* asking and posing questions, searching, being able to look deeper into ideas, and the wanting to know more about something.

6.8 Creative Thinking in Mathematics

Creativity is an attitude of mind which is encouraged by openness of thinking, willingness to work with conflicting ideas and not to have the solutions immediately, eagerness to learn, an appreciation of the working of the unconscious and a preparedness to play with an imagination and by a readiness to stand back and question the obvious. By providing the environment where the students have full freedom of thought, able to recognize their experiences independent of external restraint is conducive for promoting creative thinking. Creativity in teaching breeds creativity in learning. Creativity is defined as involving the ability to produce work that is both novel (original, unexpected) and appropriate. Creative thinking enhances problem solving ability and it provides as a tool of learning process. There are different stages in creative thinking such a separation, incubation, illumination and verification.

What is common in a lot of school mathematics problems is that they are supposed to have only one solution. Pehkonen (1995) defined this type of problem as a closed problem. He suggested that these particular problems, which do not allow divergent thinking, are not able to enhance the quality of education, even after the introduction of new approaches. If the goal of mathematics teaching is to realize an individual's potential of mathematical creativity, it is necessary to break the habit of "knowledge delivery" from the teacher to the student, which is the conventional teacher-oriented teaching method. Mathematics teaching should be focused on the development of creative thinking where students are free to try their own original possible solutions. It means avoiding the traditional teaching method that emphasizes 'convergent thinking,' in which a student memorizes existing mathematical rules and theorems and then applies them to problems with great adroitness in order to find one exclusive solution.

Since these closed problems do not encourage students to adopt divergent thinking and reasoning, it is necessary to introduce new contexts

that allow the students to respond positively and participate actively in the learning process. They are

6.8.1 Brainstorming

It is a creative problem solving technique which is used to record maximum possible number of ideas on a defined subject. This brainstorming sequence includes sensing the problem, fact finding, ideation, and evaluation of ideas and planning for implementation.

6.8.2 Self-directed Learning

It is a systematic process in which the students take up the responsibility in collaboration with others for diagnosing their own learning needs formulating learning objectives planning and engaging in a sequence of learning experience.

6.8.3 Discovery Learning

This helps the learner to achieve his own understanding of mathematics by successfully solving a carefully structured sequence of problems. Here the student discovers the knowledge independently by experimentation in exploration instead of being directly presented with a content to be learnt.

6.8.4 Guided Discovery

Here the students are not acquainted with facts rather they are made to investigate or discover the facts. The teacher only provides suitable direction to produce independent solutions to problems.

6.8.5 Encouraging Intrinsic Motivation

Creativity is developed through higher levels of intrinsic motivation. The strategies which may influence intrinsic motivation are challenge, freedom, resources and supervisory encouragement.

6.8.6 No Over Controlling

Instead of dictating the activities they should engage in, the teachers let the students select their interests and support their inclinations.

6.8.7 Fostering Flexible and Playful Thinking

Creative thinkers are flexible and play with the problems which give rise to a paradox. The considerations here are

- Being open to alternate solutions

- Practicing creativity by regularly engaging the students in activities that encourage flexible thinking
- Using multiple senses when seeking solutions by thinking in terms of the five basic senses, the kinaesthetic sense and visualizing how the solution must look.
- Playing the roles of explorer, artist, judge and lawyer

6.8.8 Questioning

The questions that elicit many answers can be put whole teaching mathematics at any level in the classroom. The students should be confronted with open ended and divergent questions.

6.8.9 Encouraging Lateral Thinking

Lateral thinking requires the students to deviate from the conventional style of vertical thinking. It involves the flexibility of being able to change ones perception of the situation to insight full situations.

6) The relation of Creativity to Problem solving and Problem Posing in Mathematics.

1. Mathematics creativity refers to discovering new connections as a result of formal changes to things that already exists.
2. Asking original questions to solve problems and presenting solutions from various viewpoints have also been referred to as mathematics creativity.
3. Mathematics creativity as two different products they are cognitive processes and result-oriented endeavors.
4. Creativity is in the nature of problem posing that is creating a problem is a creative activity.
5. It can be determined by an original solution to a problem that no one has solved before.
6. The three criteria for math creativity are fluency, flexibility and originality.

7. Problem solving is the process of understanding the problem which is the first stage of creative process.
8. Problem posing is the process of examining the problems, analyzing them and writing up using their own words.
9. It is proved that geometry has been a proper field for showing more than one way to solve a problem and then they assessed their subject's geometry knowledge and creativity using a geometry problem.
10. For example, if a student reaches a solution using a different method of others, he or she has a higher level of creativity than do others. In general problem posing, mathematics creativity and problem solving have common characteristics.